A Boundary Element Method for the Analysis of

Thin Piezoelectric Solids

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Abstract

Piezoelectric films or coatings are applied widely as sensors and actuators in smart materials and micro-electro-mechanical systems (MEMS). Analysis of these delicate thin piezoelectric solids are very important for the design and evaluations of such advanced materials. In this thesis, based on the conventional boundary integral equation (CBIE) formulation for piezoelectricity, an efficient, reliable, convenient and consistent boundary element method (BEM) is developed for the analysis of two dimensional (2-D) thin piezoelectric solids with any small thickness. An efficient analytical method is developed to deal with the nearly-singular integrals in the CBIE for 2-D thin structures. The nearly-singular integrals, which are line integrals for 2-D problems and arise when two boundary curves are close to each other, are transformed into summations of integrals containing fractions of polynomials. For the test problems studied, very promising results are obtained with the thickness-to-length ratios in the order of $10^{-6}$, which is sufficient for modeling most thin piezoelectric films and coatings in the micro-scale, as applied in smart materials and MEMS.
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Chapter 1. Introduction

1.1 The Boundary Element Method (BEM)

The Boundary Element Method (BEM) is a numerical method to solve the boundary value problem by the boundary integral equation (BIE). Since the potential problem was first formulated in terms of a direct BIE and solved numerically by Jaswon [1] about 40 years ago, extensive research effort has been made in BIE formulations of problems in mechanics and their numerical solution schemes [2-8]. Today the BIE/BEM has gained a great deal of application in many fields of computational mechanics, such as the wave propagation, heat transfer, diffusion and convection, fluid flow, fracture mechanics, electric problems, geomechanics, plates and shells, inelastic problems, contact problems, design sensitivity and optimization and inverse problems [8-11].

1.2 The BEM and Piezoelectric Materials

Piezoelectric materials have been used widely as sensors and actuators in smart materials or structures because they have many desirable properties [12-14]. Simulations of piezoelectric solids, on the other hand, are very challenging because of the anisotropy in piezoelectric materials, coupling of elastic and electric fields and thinness of the piezoelectric devices (for example, the thickness of sensors/actuators is in the range of a few μm to a few hundred μm). To add to the level of difficulty, the simulation of piezoelectric sensors and actuators demands high accuracy, because they are very delicate electromechanical devices. To ensure the highest possible accuracy in the analysis, accurate modeling and analysis have
to be employed, especially as stress analysis for durability assessment has become an important issue with the increasing applications of piezoelectric materials.

In the realm of 3-D analysis, the boundary integral equation/boundary element method (BIE/BEM) pioneered in the early work [2] for elasticity problems, has been demonstrated to be a viable alternative to the finite element method (FEM) for many problems, due to its features of surface-only discretization and high accuracy in stress and fracture analyses [8-11]. Another advantage of the BIE/BEM, which was recognized only in recent years, is its high accuracy and efficiency in handling thin-body problems, such as thin shell-like structures, layered structures (multi-coatings or thin films), thin voids or open cracks [15-22]. It has been demonstrated that the BIE/BEM can handle the various thin-body problems very effectively, regardless of the thinness of the structures or voids, or non-uniform thickness, as long as the nearly-singular integrals are computed accurately [17, 18, 23]. Much fewer boundary elements can be used to solve these problems for which the number of required finite elements is at least two-orders larger to achieve the same accuracy in stress analysis [17-20]. Considering the fact that the piezoelectric sensors and actuators are often made in thin shapes (films or patches), the BIE/BEM with thin body capabilities has the potential to provide a very efficient and accurate tool in the analysis of such piezoelectric materials.

There have been increasing research efforts in the analysis of piezoelectric materials by the BEM in recent years, as the advantage of the BEM for such analysis is being recognized. For piezoelectric solids without defects, Meric and Saigal [24] derived integral formulations for shape sensitivity analysis of 3-D piezoelectric solids. The formulations were tested on a one-dimensional problem. Lee and Jiang [25-27] developed the first BIE formulation for the piezoelectric solids. The 2-D BEM was implemented and tested for an
infinite piezoelectric medium with a cylindrical hole under mechanical and electric loads [27]. Lu and Mahrenholtz [28] derived a variational BEM formulation for piezoelectric solids, which yields symmetric matrices. However, no numerical implementation and examples were given in [28]. A 3-D BEM for piezoelectric solids was first developed by Chen and Lin [29]. The BEM formulation was based on the fundamental solutions derived earlier by Chen [30, 31] for 3-D piezoelectric solids. The numerical examples using linear elements on a piezoelectric cube and a spherical cavity were presented in [29]. Wang [32] derived the explicit expressions of the 2-D fundamental solutions for piezoelectric materials. Dunn and Wienecke [33] also derived the closed-form expressions for the fundamental solution for transversely isotropic piezoelectric solids. Hills and Farris [34] applied the quadratic (eight-node) boundary elements for 3-D piezoelectric bodies and tested their approach on the cube and spherical void problems. Ding et al. [35, 36] derived the fundamental solutions in terms of harmonic functions and developed the BEM with several test cases for 2-D [35] and 3-D problems [36]. Recently, Jiang [37] derived the fundamental solutions and the BIE for 3-D time-dependent thermo-piezoelectricity. No numerical examples were given in [37] for this very complicated case.

For piezoelectric solids with defects (various voids and cracks), Xu and Rajapakse [38] studied the influence and interactions of various holes in 2-D piezoelectric media using a coupled BEM. Zhao et al. [39, 40] derived the 3-D fundamental solutions and the BIE for a penny-shaped crack in a piezoelectric solid. Pan [41] recently presented a detailed study on cracks in 2-D piezoelectric media using the BEM. Both the conventional BIE and a hypersingular BIE (traction BIE) were employed in [41] to handle the possible degeneracy of the BIEs for crack problems. The degeneracy of the conventional BIE is assumed for crack
problems but not justified analytically in [41]. Numerical results in [41] show excellent agreement between the BEM and the analytical solutions. Recently, Qin [42] studied the interactions of cracks in a piezoelectric half-plane and under thermal loading using the BEM.

All the above results have clearly demonstrated the accuracy and efficiency of the BEM, especially in stress and fracture analyses, for single and bulky piezoelectric materials.

1.3 The structure of This Thesis

In Chapter 2, the BIE formulation for piezoelectric solids is visited first. A weakly-singular form of the piezoelectric BIE is developed using the identities derived earlier, which can eliminate the calculation of any singular integrals in the discretizations of the piezoelectric BIE using the BEM. Then the degeneracy issues with the piezoelectric BIE for thin shapes has been discussed, An example has been used to test it.

In Chapter 3, an efficient analytical method is developed to deal with the nearly-singular integrals in the CBIE for 2-D thin structure. The nearly-singular integrals, which are line integrals for 2-D problems and arise when two boundary curves are close to each other, are transformed into the summation of polynomial fraction integral at [-1,1]. In addition, a new nonlinear coordinate transformation is developed for nearly weakly-singular integrals to further increase the numerical accuracy. For the test problems studied, very promising results are obtained when the thickness to length ratio is in the order of $10^{-6}$, which is sufficient for modeling most thin structures in the micro-scales.

The thesis concludes with Chapter 4 on some discussions and conclusions. The fundamental solutions for 2-D piezoelectric solids are given in detail in the Appendix A and the developed C++ code is listed in Appendix B.
2.1 Governing Equations in Piezoelectricity

Consider a piezoelectric solid occupying a 3-D domain $V$ with the boundary $S$, Figure 2.1. The basic equations governing the elastic and electric fields in a linear piezoelectric material can be summarized in the following (see, e.g., Refs. [12, 26, 29]) (index notation is used here).

*Equilibrium Equations:*

\[ \sigma_{ij,j} + f_i = 0, \]

\[ D_{i,i} - q = 0, \]

where $\sigma_{ij}$ is the stress tensor, $f_i$ the body force vector per unit volume, $D_i$ the electric displacement vector and $q$ the intrinsic electric charge per unit volume.

*Constitutive Equations:*

\[ \sigma_{ij} = C_{ijkl} s_{kl} - e_{ij}^{kl} E_k, \quad \text{(converse effect)} \]  \hspace{1cm} (2.3)

\[ D_i = e_{ik}^{kl} s_{kl} + \varepsilon_{ik} E_k, \quad \text{(direct effect)} \]  \hspace{1cm} (2.4)

where $s_{kl}$ is the strain tensor, $E_k$ the electric field, $C_{ijkl}$ the elastic modulus tensor measured in a constant electric field, $e_{ij}^{kl}$ the piezoelectric tensor and $\varepsilon_{ij}$ the dielectric tensor measured at constant strains.

*Strain and Electric Fields:*

\[ s_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \]

\[ E_i = -\phi_{,i}, \]

\[ s_{y} = \frac{1}{2} (u_{i,j} + u_{j,i}), \]

\[ E_i = -\phi_{,i}, \]
where \( u_i \) is the elastic displacement vector and \( \phi \) the electric potential.

**Boundary Conditions (BCs):**

\[
t_i = \sigma_i n_j = \bar{t}_i, \quad \text{on } S_i, \quad \quad u_i = \bar{u}_i, \quad \text{on } S_u; \quad \quad \text{(mechanical BCs)} \quad (2.7)
\]

\[
\omega = - D_i n_i = \bar{\omega}, \quad \text{on } S_\omega, \quad \quad \phi = \bar{\phi}, \quad \text{on } S_\phi; \quad \quad \text{(electric BCs)} \quad (2.8)
\]

where \( t_i \) is the traction, \( \omega \) the surface charge, \( n_i \) the unit outward normal vector (Figure 2.2) and the barred quantities indicate given values. Note that the boundary \( S = S_i \cup S_u = S_\omega \cup S_\phi \).

Equations (2.1-6) under boundary conditions (2.7-8) form the complete mathematical description of the coupled elastic and electric fields in a general anisotropic piezoelectric solid. For an isotropic elastic material, there is no coupling of the elastic and electric fields, that is, the piezoelectric tensor \( e_{\alpha k} = 0 \). In this case, Equations (2.1-6) will be decoupled between the two fields, yielding the usual elasticity equations and a Poisson’s equation for the electric potential \( \phi \).

The boundary integral equation formulation in weakly-singular forms for piezoelectric solids has been derived in the recent paper [43]. The degeneracy issues associated with the BIE for piezoelectric thin bodies or thin shapes (such as cracks) have also been discussed in [43]. For completeness of this thesis, these theoretical results are given in the following sections.

**2.2 Weakly-Singular BIE Formulation for Piezoelectricity**

We first note the following generalized Green’s identity, or reciprocal work theorem, for the piezoelectric solids [43]:

\[
\int_{S_i} t_i \cdot n_i \, dS - \int_{S_u} u_i \cdot n_i \, dS - \int_{S_\omega} \omega \cdot n_i \, dS - \int_{S_\phi} \phi \cdot n_i \, dS = 0
\]
\[
\int_S t_i u^*_i dS + \int_V f_i u^*_i dV + \int_S \omega^*_i \phi dS + \int_V q^*_i \phi dV
\]
\[
= \int_S t_i u_i dS + \int_V f_i u_i dV + \int_S \omega^*_i \phi dS + \int_V q^*_i \phi dV,
\]
(2.9)
in which \( u_i, t_i, \phi, \omega, \ldots \), and \( u^*_i, t^*_i, \phi^*, \omega^*, \ldots \) are two sets of admissible solutions satisfying equations (2.1-8).

Here, if we choose the field
\[
u^*_j = U_j, \quad t^*_j = T_j, \quad \phi^* = \Phi, \quad \omega^* = \Omega, \quad f^*_j = \delta (P, P_o), \quad \text{and} \quad q^* = 0,
\]
to be the fundamental solution due to a unit force, while the field \( u_j, t_j, \phi, \omega \) and \( q \) to be the solution satisfying equations (2.1-8), we have from identity (2.9),
\[
\int_S t_j U_j dS + \int_V f_j U_j dV + \int_S \Phi \delta dS
\]
\[
= \int_S T_j u_j dS + \int_V \delta (P, P_o) u_j dV + \int_S \omega \Phi dS + \int_V q \Phi dV.
\]
(2.10)
Using the property of the \( \delta \)-function, we have the following representation integral for the displacement field,
\[
u_i (P_o) = \int_S U_j (P, P_o) t_j (P) dS (P) - \int_S T_j (P, P_o) u_j (P) dS (P)
\]
\[
- \int \Phi_i (P, P_o) \omega (P) dS (P) + \int \Omega_i (P, P_o) \phi (P) dS (P)
\]
\[
+ \int U_j (P, P_o) f_j (P) dV (P) - \int \Phi_i (P, P_o) q (P) dS (P), \quad \forall P_o \in V,
\]
(2.11)
in which \( i, j = 1, 2, 3 \).

Similarly, if we choose, in identity (2.9),
\[
u^*_j = U_{4j}, \quad t^*_j = T_{4j}, \quad \phi^* = \Phi_4, \quad \omega^* = \Omega_4, \quad f^*_j = 0, \quad \text{and} \quad q^* = \delta (P, P_o),
\]
to be the fundamental solution due to a unit charge, we obtain the following representation integral for the electric potential field,
\[
- \phi (P_o) = \int_S U_{4j} (P, P_o) t_j (P) dS (P) - \int_S T_{4j} (P, P_o) u_j (P) dS (P)
\]
\[
- \int \Phi_4 (P, P_o) \omega (P) dS (P) + \int \Omega_4 (P, P_o) \phi (P) dS (P)
\]
\[
+ \int U_{4j} (P, P_o) f_j (P) dV (P) - \int \Phi_4 (P, P_o) q (P) dS (P), \quad \forall P_o \in V,
\]
(2.12)
in which \( j = 1, 2, 3 \).

From the two representation integrals (2.11) and (2.12), we can clearly identify the similarities as compared with the elasticity case and the coupling between the displacement and electric fields. To simplify the notation to make it easier for the numerical implementation, we adopt the following matrix notation [35],

\[
\begin{bmatrix}
u
\end{bmatrix},
\begin{bmatrix}
t
\end{bmatrix},
\begin{bmatrix}
f
\end{bmatrix},
\begin{bmatrix}
\Omega
\end{bmatrix},
\begin{bmatrix}
\Phi
\end{bmatrix}
\]

\[
\begin{bmatrix}
U_{11} & U_{12} & U_{13} & \Phi_1 \\
U_{21} & U_{22} & U_{23} & \Phi_2 \\
U_{31} & U_{32} & U_{33} & \Phi_3 \\
U_{41} & U_{42} & U_{43} & \Phi_4 \\
t_1 & t_2 & t_3 & -\omega
\end{bmatrix},
\begin{bmatrix}
T_{11} & T_{12} & T_{13} & \Omega_1 \\
T_{21} & T_{22} & T_{23} & \Omega_2 \\
T_{31} & T_{32} & T_{33} & \Omega_3 \\
T_{41} & T_{42} & T_{43} & \Omega_4
\end{bmatrix}
\]

Then, the representation integrals (2.11) and (2.12) can be combined to yield,

\[
\mathbf{u}(\mathbf{P}_o) = \int_\mathbf{S} \mathbf{U}(\mathbf{P}, \mathbf{P}_o) \mathbf{t}(\mathbf{P}) d\mathbf{S}(\mathbf{P}) - \int_\mathbf{S} \mathbf{T}(\mathbf{P}, \mathbf{P}_o) \mathbf{u}(\mathbf{P}) d\mathbf{S}(\mathbf{P})
\]

\[
+ \int_\mathbf{V} \mathbf{U}(\mathbf{P}, \mathbf{P}_o) \mathbf{b}(\mathbf{P}) d\mathbf{V}(\mathbf{P}), \quad \forall \mathbf{P}_o \in \mathbf{V},
\]

where \( \mathbf{u}, \mathbf{t} \) and \( \mathbf{b} \) can be called the generalized (or extended) displacement, traction and body force vectors, respectively; and \( \mathbf{U} \) and \( \mathbf{T} \) the generalized displacement and traction kernels, respectively.

We have the following identity for the piezoelectric fundamental solution [43]:

\[
\int_\mathbf{S} \mathbf{T}(\mathbf{P}, \mathbf{P}_o) d\mathbf{S}(\mathbf{P}) = \begin{cases}
-\mathbf{I}, & \forall \mathbf{P}_o \in \mathbf{V}, \\
-\frac{1}{2}\mathbf{I}, & \forall \mathbf{P}_o \in \mathbf{S} \text{ (smooth)}, \\
\mathbf{0}, & \forall \mathbf{P}_o \in \mathbf{E},
\end{cases}
\]

where \( \mathbf{I} \) is a \( 4 \times 4 \) identity matrix.

Therefore, for the integral with the strongly singular kernel \( \mathbf{T} \) in (2.14), we have
\[
\int_S \mathbf{T}(P, P_o) \mathbf{u}(P) dS(P) = \int_S \mathbf{T}(P, P_o) [\mathbf{u}(P) - \mathbf{u}(P_o)] dS(P) + \int_S \mathbf{T}(P, P_o) dS(P) \mathbf{u}(P_o)
\]
\[
= \int_S \mathbf{T}(P, P_o) [\mathbf{u}(P) - \mathbf{u}(P_o)] dS(P) - \mathbf{u}(P_o), \quad \forall P_o \in V,
\]
(2.16)

by using identity (2.15) (cf., the potential and elasticity cases [44-46]).

Substituting result (2.16) into (2.14), and letting the source point \( P_o \) go to the boundary \( S \), we obtain the following weakly-singular form of the boundary integral equation in piezoelectricity,

\[
\int_S \mathbf{T}(P, P_o) [\mathbf{u}(P) - \mathbf{u}(P_o)] dS(P) = \int_S \mathbf{U}(P, P_o) \mathbf{t}(P) dS(P)
\]
\[
+ \int_v \mathbf{U}(P, P_o) \mathbf{b}(P) dV(P), \quad \forall P_o \in S,
\]
(2.17)

for a finite domain (interior problem). There are no jump terms arising from the limit process as the source point \( P_o \) goes to the boundary \( S \), since all integrals involved are at most weakly singular, e.g., of order \( O(1/r) \) for 3-D problems, after the regularization shown in (2.17).

Similarly, for an infinite domain (exterior problem), we can establish the following weakly-singular form of the boundary integral equation

\[
\mathbf{u}(P_o) + \int_S \mathbf{T}(P, P_o) [\mathbf{u}(P) - \mathbf{u}(P_o)] dS(P) = \int_S \mathbf{U}(P, P_o) \mathbf{t}(P) dS(P)
\]
\[
+ \int_v \mathbf{U}(P, P_o) \mathbf{b}(P) dV(P), \quad \forall P_o \in S.
\]
(2.18)

The weakly-singular BIE (2.17) or (2.18) for piezoelectric solids has several advantages, compared with the following singular BIE in the literature:

\[
\mathbf{C}(P_o) \mathbf{u}(P_o) + \int_S \mathbf{T}(P, P_o) \mathbf{u}(P) dS(P) = \int_S \mathbf{U}(P, P_o) \mathbf{t}(P) dS(P)
\]
\[
+ \int_v \mathbf{U}(P, P_o) \mathbf{b}(P) dV(P), \quad \forall P_o \in S,
\]
(2.19)

where \( \mathbf{C} \) is a coefficient matrix depending on the smoothness of \( S \) at \( P_o \) (see next section).

First, there are no singular integrals in the weakly-singular BIE and its discretization leads directly to the conclusion that the diagonal terms can be determined by summing the
off-diagonal terms for the matrix involving the singular kernel $T$ [45]. Second, by employing the identity for the piezoelectric fundamental solution, we do not have to evaluate any jump terms explicitly in deriving the weakly-singular BIE (2.17) or (2.18). Evaluations of the jump terms are required in deriving the singular BIE (2.19) as used in the literature. For general 3D piezoelectric solids, the fundamental solution is not available in explicit form and the jump terms have been assume to be the same as in the elasticity BIE, without sufficient justifications, in the literature. The use of the weakly-singular BIE can avoid the confusions caused by this lack of justifications. Third, regularizing the singular integrals in the BIEs has become a standard approach to dealing with the singular integrals in the BEM [47, 48]. The weakly-singular nature of the piezoelectric BIE is quite general, as in the cases for potential and elastostatic problems [44-46]. Not only can the various strongly singular (conventional) BIEs be recast in weakly-singular forms, but also can the hypersingular BIEs be written in weakly-singular forms, by employing the various identities for the fundamental solutions [45, 46] or through other means.

2.3. Degeneracy Issues with the Piezoelectric BIE for Thin Shapes

In this section, it is shown that the piezoelectric BIE does degenerate when applied to the two opposing surfaces of a crack, but does not degenerate when applied to the two surfaces of a thin shell-like structure [43].

2.3.1. Degeneracy of the Piezoelectric BIE for Crack Problems

Using the matrix notation and following the steps in Ref. [17] for the elasticity BIE, we can derive the following two equations in the limit as the distance $h \to 0$, for a crack-like problem (Figure 2.3) [43]:

$$
\begin{align*}
\end{align*}
$$
\[
D^+(u^+ - u^-) + \frac{1}{2}(u^+ + u^-) = B^+(t^+ + t^-), \quad \text{from } P_o \in S^+; \quad (2.20)
\]
\[
D^+(u^+ - u^-) - \frac{1}{2}(u^+ + u^-) = B^+(t^+ + t^-), \quad \text{from } P_o \in S^-; \quad (2.21)
\]

where \( u^+, t^+, u^-, t^- \) are the generalized displacement and traction vectors, on \( S^+ \) and \( S^- \), respectively. \( D^+ \) and \( B^+ \) are matrices or integral operators associated with the \( U \) and \( T \) kernels, respectively. Equations (2.20) and (2.21) are exactly the same equation. Therefore, the piezoelectric BIE does degenerates when applied to cracks.

2.3.2. Non-Degeneracy of the Piezoelectric BIE for Thin Piezoelectric Shells

Applying BIE (2.19) on the two surfaces of a thin piezoelectric shell (an interior-like problem, Figure 2.4), we can obtain the following two equations in the limit as the thickness \( h \to 0 \) [43]:
\[
D^+(u^+ - u^-) + \frac{1}{2}(u^+ + u^-) = B^+(t^+ + t^-), \quad \text{from } P_o \in S^+; \quad (2.22)
\]
\[
D^+(u^+ - u^-) - \frac{1}{2}(u^+ + u^-) = B^+(t^+ + t^-), \quad \text{from } P_o \in S^-; \quad (2.23)
\]

which are two distinctive equations no matter how thin the shell is, as long as the piezoelectric shell is under realistic boundary conditions (e.g., not constrained on the entire boundary \( S \); see discussions in [17]). Therefore, the piezoelectric BIE does not degenerate when applied to thin shells.

2.4 Numerical Examples

To demonstrate that the piezoelectric BIE does degenerate for crack problems, a test problem is considered in this section. A multi-domain BEM approach to remedy this
The degeneracy problem in using the conventional piezoelectric BIE for crack problems is also presented.

A two-dimensional BEM based on the piezoelectric BIE (2.18) without body forces and charges, is implemented using quadratic (three-node line) elements. Figure 2.5 shows a square piezoelectric medium (PZT-4, under plane strain condition) with an elliptical hole at the center. The square domain is sufficiently large compared with the hole ($L/b = 20$) so that the analytical solution in Ref. [49] for an infinite domain can be used to validate the BEM solutions. The materials constants of PZT-4 are as given in Ref. [27] in which only the circular hole case is considered. The model is under uniform tension in the $x_1$ direction. The elliptical holes are formed by starting with a circular hole and scaling it in the $x_1$ direction (axis $a$).

Figures 2.6, 2.7 and 2.8 show the results of the first principal stress, total mechanical displacement and electric displacement at the nodes on the holes, respectively, for three values of the ratio $a/b$, by the piezoelectric BEM and the analytical solution [49] (Note that the angle $\theta$ is measured on the original circular hole in all the cases, Figure 2.5). It is shown that the BEM results using only 44 elements are in excellent agreement with the exact solution for all the three quantities in the three cases studied ($a/b = 1.0, 0.5$ and $0.05$).

When the ratio $a/b$ is further reduced and the elliptical hole becomes an open crack, many more elements on the edge of the hole are needed in order to obtain possible converged BEM results. Figure 2.9 shows the BEM results for the mechanical displacement at the hole when $a/b = 0.01$ with increasing numbers of elements. It is observed that only when the elliptical hole is discretized using 180 elements (a total of 204 elements), do the BEM results converge to the analytical solution. However, when the case $a/b = 0.001$ is studied, even the
finest BEM mesh (204 elements) cannot provide converged results, as shown in Figure 2.10 for the electric displacement result. Moreover, the symmetry in the BEM results with respect to the crack tip ($\theta = 90^\circ$) is also lost (Figure 2.10). This is a clear indication of the degeneracy of the piezoelectric BIE/BEM for crack problems, as predicted by the theory in section 3.2.

As discussed in Refs. [17, 18] and by many others, there are two difficulties when the electricity BIE/BEM is applied to a thin void (or open crack) with increasingly smaller opening. The first difficulty is in dealing with the nearly-singular integrals when the source point is on one surface and the integration on the opposite surface of the crack. The other difficulty is the degeneracy of the BIE when applied to true (zero-opening) cracks. These two difficulties also exist in the piezoelectric BIE/BEM when applied to a thin void in a piezoelectric material, as has been proved and demonstrated in this chapter. Increasing the number of elements on the two faces of the open crack (thus decreasing the element sizes) can alleviate the difficulty in computing nearly-singular integrals, although it is not the efficient way to deal with nearly-singular integrals in the BEM [17, 18]. This is why good BEM results are obtained with a large number of elements in the case when $a/b = 0.01$ (Figure 2.9). However, increasing the number of elements will not help in easing the difficulty of the BIE/BEM degeneracy for crack problems. This is why the BEM results deteriorate even with the large number of elements in the $a/b = 0.001$ case, which is closer to a true crack (Figure 2.10). Alternative BIE formulations or different BEM modeling techniques are needed in order to tackle the crack problems.

Next, the multi-domain BEM approach to crack problems is demonstrated in the context of piezoelectric BIE. The main idea in the multi-domain BEM for crack problems (see, e.g., Ref. [8]) is to introduce auxiliary interfaces in the domain, starting from the crack
tips to the outer boundary of the domain so that the original single domain is divided into two (or several, if needed). Then the conventional BIE is applied to each domain and the two systems of equations are coupled together through the use of interface conditions (e.g., continuity of displacements and equilibrium of stresses). In this way, the degeneracy difficulty, and the nearly-singular integrals in some cases, in the BEM for crack problems are avoided since the two equations on the two opposing crack faces are now from different domains. The disadvantage of using this multi-domain approach to crack problems is that the auxiliary interfaces introduced could be large and hence the problem size could be much larger. Nevertheless, it is a simple, straightforward approach to crack problems by using only the conventional BIE.

Figure 2.11 shows the division of the piezoelectric medium, considered earlier (see Figure 2.5), into two subdomains. Figures 2.12-14 show the stress, displacement and electric displacement, respectively, on the edge of the hole (in fact, an open crack) when \( a/b = 0.001 \) (Figure 2.5), using the multi-domain BEM as compared with the single-domain BEM with the same mesh on the hole and the outer boundary. Additional 20 elements are employed on the two interface lines (Figure 2.11) for the multi-domain BEM. As shown and discussed earlier (Figure 2.10), the single-domain BEM results depart dramatically from the analytical solutions for all the three quantities, due to the degeneracy of the BIE in this open-crack case. However, the multi-domain BEM results agree very well with the analytical solutions with a relatively small number of elements, as expected. Note that the BEM is able to capture the strong singularity of the stresses near the crack tip as shown in Figure 2.12. Also note that the hoop stress on the edge of the hole is compressive in most regions in this crack-like case, therefore the first principal stress is zero, except for the regions near the crack tips. The
multi-domain BEM results for the electric displacement (Figure 2.14) is less accurate in the small region of the crack tip. This is due to the rapid oscillation of the field when $a/b = 0.001$ as shown by the analytical solution.

The multi-domain piezoelectric BEM based on the conventional BIE, as demonstrated in the above, can provide reasonably good results for the analysis of crack-like problems. However, the more efficient and accurate way to handle crack problems is to apply the piezoelectric hypersingular BIE [41].

### 2.5 Conclusion

In this chapter the whole BEM process for piezoelectric problems is discussed. An example is used for testing the Degeneracy of the Piezoelectric BIE for Thin Piezoelectric Cracks.

An important issue deserving mention here is that when the source node $P_0$ is near to the element, the integral is nearly singular. In the next chapter, the evaluation of the nearly-singular integrals by a newly developed method will be discussed in details.
Figure 2.1. Domain $V$ in $\mathbb{R}^3$ with boundary $S$ (exterior domain $E = \mathbb{R}^3 - (V \cup S)$).

Figure 2.2. Boundary $S$ of a thin-shape: $S = S^+ \cup S^-$. 
Figure 2.3. Boundary and normal for a crack (an exterior problem).

Figure 2.4. Boundary and normal for a thin piezoelectric shell (an interior problem).
Figure 2.5. An elliptical hole in a square piezoelectric medium under tension (plane strain condition).
Figure 2.6. The first principal stress ($\sigma_1''$) on the edge of the holes (number of boundary elements $M = 44$).
Figure 2.7. The total mechanical displacement ($|u|$) on the edge of the holes (number of boundary elements $M = 44$).
Figure 2.8. The magnitude of electric displacement ($|D|$) on the edge of the holes (number of boundary elements $M = 44$).
Figure 2.9. The total mechanical displacement on the edge of the hole when $a/b = 0.01$ ($M =$ number of boundary elements applied).
Figure 2.10. Degeneracy of the piezoelectric BIE/BEM: Results for the magnitude of electric displacement on the edge of the hole when $a/b = 0.001$ ($M =$ number of boundary elements applied).
Figure 2.11. A multi-domain BEM approach to the crack-like problems.
Figure 2.12. The first principal stress on the edge of the hole when $a/b = 0.001$ using the multi-domain BEM.
Figure 2.13. The total mechanical displacement on the edge of the hole when $a/b = 0.001$ using the multi-domain BEM.
Figure 2.14. The magnitude of electric displacement on the edge of the hole when $a/b = 0.001$ using the multi-domain BEM.
Chapter 3. Analysis of 2-D Thin Piezoelectric Solids

3.1 Introduction

In recent years, piezoelectric materials have been used widely as sensors and actuators in smart materials and in micro-electro-mechanical systems (MEMS), because piezoelectric materials possess many desirable properties [12-14]. Analysis of the piezoelectric sensors and actuators is, however, very difficult because they are usually made in the forms of thin films or coatings attached to elastic substrates. Detailed stress analysis for durability assessment of such materials is challenging for any numerical methods based on the piezoelectric plate or shell theories, especially in evaluating interface stresses which exhibit singularities near the edges of the films (see, e.g., Ref. [50]). To ensure the highest possible accuracy in the stress or fracture analysis of the delicate piezoelectric films and coatings, accurate 2-D or 3-D models based on the piezoelectricity theory should be employed.

In the last decade, there have been increasing efforts in the research on modeling piezoelectric materials using the boundary integral equation/boundary element method (BIE/BEM) based on the piezoelectricity theory. For example, Lee and Jiang [25-27] developed the BEM formulation for piezoelectric solids and tested their method on a 2-D infinite piezoelectric medium with a cylindrical hole [27]. A 3-D BEM for piezoelectric solids was developed by Chen and Lin [29], based on the fundamental solutions derived earlier [30, 31] for 3-D piezoelectric solids. The numerical examples using linear elements on a piezoelectric cube and a spherical cavity were presented in [29]. Hills and Farris [34] applied the quadratic (eight-node) boundary elements for 3-D piezoelectric bodies and tested their approach on the cube and spherical void problems. Ding et al. [35, 36] derived the
fundamental solutions in terms of harmonic functions and developed the BEM with several test cases for 2-D [35] and 3-D problems [36]. Recently, Jiang [37] derived the fundamental solutions and the BIE for 3-D time-dependent thermo-piezoelectricity. For piezoelectric solids with defects (various voids and cracks), Xu and Rajapakse [38] studied the influence and interactions of various holes in 2-D piezoelectric media using a coupled BEM. Zhao and et al. [39, 40] derived the 3-D fundamental solutions and the BIE for a penny-shaped crack in a piezoelectric solid. Pan [41] recently presented a detailed study on cracks in 2-D piezoelectric media using the BEM. Both the conventional BIE and a hypersingular BIE (traction BIE) were employed in [41] to handle the degeneracy of the BIEs for crack problems. Recently, Qin [42] studied the cracks in a piezoelectric half-plane and under thermal loading with the BEM.

All the above results using the BEM have clearly demonstrated the accuracy and efficiency of the piezoelectric BEM, especially in stress and fracture analyses, for single and bulky piezoelectric materials. However, no piezoelectric BEM has been attempted to analyze thin piezoelectric materials, such as piezoelectric films and coatings. Applications of the piezoelectric BEM to thin piezoelectric structures face two crucial questions: 1) Does the piezoelectric BIE degenerate when applied to shell-like piezoelectric structures? 2) How to deal with the nearly-singular integrals existing in the piezoelectric BIE when applied to shell-like and crack-like problems? The first question has been addressed in a recent paper [43] which proves that the piezoelectric BIE does not degenerate when applied to thin piezoelectric shell-like structures, contrary to the case of modeling cracks in piezoelectric materials using the BIE. However, according to the best knowledge of the authors, no work has been reported
in the literature to address the second question regarding the nearly-singular integrals in the piezoelectric BIE when applied to shell-like piezoelectric structures.

In the context of elastic structures or materials, it has been demonstrated that the BIE/BEM with thin-body capabilities can handle various thin shell-like problems very effectively, regardless of the thinness of the structures, or non-uniform thickness, as long as the nearly-singular integrals are computed accurately. Numerous examples of applying the BEM based on the elasticity theory to both 2-D and 3-D thin shell-like structures, including thin elastic layered films, coatings, interphases in composites, with or without interface cracks, can be found in Refs. [17-22, 51]. These studies have shown that accurate, efficient, and yet stable displacement and stress results can be obtained using the BEM for the analysis of the thin structures or materials with the thickness-to-length ratios in the range of $10^{-6}$ to $10^{-9}$, once the nearly-singular integrals are computed accurately. The semi-analytical approach for 3-D case and the analytical approach for 2-D case developed in the above mentioned work to deal with the nearly-singular integrals in the elasticity BIEs can be extended to the 2-D piezoelectric BIE with no new conceptual challenges, although it is much more difficult for the piezoelectric BIE case.

In this chapter, a piezoelectric BEM is developed for analyzing 2-D thin piezoelectric structures with very small thickness-to-length ratios. The nearly-singular integrals (line integrals for 2-D problems) are transformed into integrals containing summations of several polynomial fractions, which can be computed analytically. For the test problems on piezoelectric films and coatings, very promising BEM results with only a few boundary elements are obtained when the thickness-to-length ratio is as small as $10^{-6}$. This is sufficient
for modeling most thin piezoelectric films and coatings in various applications such as smart materials and MEMS.

### 3.2. The BIE Formulation for 2-D Piezoelectricity

For two-dimensional problems in analyzing piezoelectric materials, the representation integral is given as follows (vector notation is used here; see, Refs. [35, 43]):

\[
\mathbf{u}(P_o) = \int_S \mathbf{U}(P, P_o) \mathbf{t}(P) dS(P) - \int_S \mathbf{T}(P, P_o) \mathbf{u}(P) dS(P)
\]

\[+ \int_B \mathbf{U}(P, P_o) \mathbf{b}(P) dV(P), \quad \forall P_o \in V, \tag{3.1}\]

in which,

\[
\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\
\mathbf{u}_2 \\
-\phi \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} \mathbf{t}_1 \\
\mathbf{t}_2 \\
-\omega \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\
\mathbf{b}_2 \\
-\mathbf{q} \end{bmatrix}, \tag{3.2}\]

\[
\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\
U_{21} & U_{22} & U_{23} \\
U_{31} & U_{32} & U_{33} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33} \end{bmatrix},
\]

are the generalized (or extended) displacement, traction and body force vectors, and the generalized displacement and traction kernels in the 2-D piezoelectric fundamental solutions (see Appendix), respectively; \(P_o\) the source point and \(P\) the field point. Note that the dimensions of matrices are 3\(\times\)3 for 2-D piezoelectric problems because of the coupling of the elastic and electric fields.

Let the source point \(P_o\) go to the boundary \(S\) in the above representation integral, we obtain the piezoelectric boundary integral equation in the traditional (singular) form:
\[
\mathbf{C}(P_o)\mathbf{u}(P_o) + \int_S \mathbf{T}(P, P_o)\mathbf{u}(P) dS(P) = \int_S \mathbf{U}(P, P_o)\mathbf{t}(P) dS(P) \\
+ \int_v \mathbf{U}(P, P_o)\mathbf{b}(P) dV(P), \quad \forall P_o \in S, \tag{3.3}
\]

where \( \mathbf{C} \) is a 3×3 coefficient matrix depending on the smoothness of \( S \) at \( P_o \).

Applying the integral identity for the piezoelectric fundamental solution developed in Ref. [43], we can obtain the following *weakly-singular form* of the boundary integral equation for piezoelectricity [43]:

\[
\int_S \mathbf{T}(P, P_o)[\mathbf{u}(P) - \mathbf{u}(P_o)] dS(P) = \int_S \mathbf{U}(P, P_o)\mathbf{t}(P) dS(P) \\
+ \int_v \mathbf{U}(P, P_o)\mathbf{b}(P) dV(P), \quad \forall P_o \in S, \tag{3.4}
\]

for a *finite* piezoelectric solid (cf., the potential and elasticity cases [45, 52]).

Before one can apply the piezoelectric BIE (3.4) to analyze thin piezoelectric structures, the following two questions must be addressed: 1) Does the piezoelectric BIE (3.4), or equivalently, BIE (3.3), degenerate or not when applied to shell-like piezoelectric structures? 2) How to deal with the nearly-singular integrals existing in the piezoelectric BIE (4) when it is applied to shell-like and crack-like problems? The first question has been addressed in the recent paper [43] (see also Chapter 2) in which it is shown for the first time that the piezoelectric BIE does *not* degenerate when applied to thin piezoelectric shell-like structures, contrary to the case of modeling cracks in piezoelectric materials using the BIE. However, the numerical difficulty in applying the piezoelectric BIE for such thin structures, that is, how to deal with the nearly-singular integrals in the piezoelectric BIE (3.4), has not been addressed in the literature so far. The next section is a first attempt to address this crucial issue.
3.3. Nearly-Singular Integrals in the Piezoelectric BIE for Thin Shapes

The numerical difficulty in applying the piezoelectric BEM based on BIE (3.4) is the nearly-singular integrals which arise in both crack-like and thin shell-like problems. The integrals in the BIE, which determine the influence matrices, contain singular kernels of order $O(1/r)$ and $O(\ln r)$ in 2-D piezoelectricity case (same as in the elasticity case), where $r$ is the distance between the source point $P_0$ and the field (integration) point $P$ (Fig. 1). When the source point is very close to, but not on an element of integration, although the kernels are regular in the mathematical sense, values of the kernels change rapidly on that element. The standard Gaussian quadrature is no longer accurate or practical in this case since a large number of integration points are needed in order to achieve the required accuracy. Even the weakly-singular form of the BIE, such as BIE (3.4), can not avoid this nearly-singular integral difficulty. As discussed in Refs. [17, 18, 23, 53] for the 2-D and 3-D elasticity BIE cases, a very accurate and yet efficient approach to deal with these nearly-singular integrals is to evaluate them analytically through some manipulations.

A similar approach to that developed for the elastostatic BIE in Refs. [17, 18, 23] for 2-D and 3-D thin shell-like structures is investigated in this chapter to regularize the nearly-singular integrals in the 2-D piezoelectric BIE. Although the case for the piezoelectric BIE is much more complicated due to the lengthy expressions for the piezoelectric fundamental solution, it is still possible to analytically evaluate the nearly-singular integrals as in the elasticity BIE case.

We first consider the integral in BIE (3.3) or (3.4) containing the singular kernel $T_{ij}$ when the source point is close to but not on the element of integration. By subtracting and
adding back a term in the following manner, the nearly-singular integral can be rewritten as (see Refs. [17, 18, 23] for details):

$$\int_{\Delta L} T_{ij}(P, P'_0) u_j(P) dL(P) = \int_{\Delta L} T_{ij}(P, P'_0) [u_j(P) - u_j(P'_0)] dL(P) + u_j(P'_0) \int_{\Delta L} T_{ij}(P, P_0) dL(P), \quad (3.5)$$

where $\Delta L$ is the line element under consideration, $P'_0$ is the closest point on the element to $P_0$ (an image point of $P_0$ on the element [17, 18, 23]). Note that the indices $i$ and $j$ run from 1 to 3 for the 2-D piezoelectric BIE case (see, Eq. (10)). As $P \to P'_0$, the term $u_j(P) - u_j(P'_0)$ has the order of $O(r')$, with $r'$ being the distance from $P'_0$ to $P$. The order of the first integral on the right-hand side of (3.5) is reduced to $O(r')/O(r)$. This integral is a nearly weakly-singular integral, which can be evaluated accurately by a nonlinear coordinate transformation developed for the 2-D elasticity case in Ref. [18]. Now we focus on the evaluation of the last integral in (3.5) containing only the singular kernel $T_{ij}$ by using an analytical method.

Let us analyze the expressions of the $T_{ij}$ kernel first. For 2-D piezoelectric problems, the fundamental solution is much more complicated and the expressions for the corresponding kernels are very lengthy as shown in the Appendix. For brevity and clarity, we only list the major expressions here and leave the complete expressions in the Appendix and the references therein. The $T_{ij}$ kernel can be expressed as (see Appendix):

$$T_{11} = (c_{11} U_{111} + c_{12} U_{122} + e_{21} U_{132}) n_1 + (c_{33} (U_{112} + U_{121}) + e_{13} U_{131}) n_2,$$

$$T_{21} = (c_{11} U_{211} + c_{12} U_{222} + e_{21} U_{232}) n_1 + (c_{33} (U_{212} + U_{221}) + e_{13} U_{231}) n_2,$$

$$T_{31} = (c_{11} U_{311} + c_{12} U_{322} + e_{21} U_{332}) n_1 + (c_{33} (U_{312} + U_{321}) + e_{13} U_{331}) n_2.$$
\[ T_{12} = (c_{12} U_{111} + c_{22} U_{122} + e_{22} U_{132}) n_2 + (c_{33} (U_{112} + U_{121}) + e_{13} U_{131}) n_1, \]
\[ T_{22} = (c_{12} U_{211} + c_{22} U_{222} + e_{22} U_{232}) n_2 + (c_{33} (U_{212} + U_{221}) + e_{13} U_{231}) n_1, \]
\[ T_{32} = (c_{12} U_{311} + c_{22} U_{322} + e_{22} U_{332}) n_2 + (c_{33} (U_{312} + U_{321}) + e_{13} U_{331}) n_1, \]
\[ T_{13} = -(e_{13} (U_{112} + U_{121}) - e_{11} U_{131}) n_1 - (e_{21} U_{111} + e_{22} U_{122} - e_{22} U_{132}) n_2, \]
\[ T_{23} = -(e_{13} (U_{212} + U_{221}) - e_{11} U_{231}) n_1 - (e_{21} U_{211} + e_{22} U_{222} - e_{22} U_{232}) n_2, \]
\[ T_{33} = -(e_{13} (U_{312} + U_{321}) - e_{11} U_{331}) n_1 - (e_{21} U_{311} + e_{22} U_{322} - e_{22} U_{332}) n_2, \]

where \( n_1 \) and \( n_2 \) are directional cosines of the normal \( n \), constants \( c_{11} \) to \( e_{22} \) are material related and \( U_{ijk} \) are the spatial derivatives of the \( U_{ij} \) kernel (see Appendix).

From the lengthy expressions listed in the Appendix, we can conclude that the \( T_{ij} \) kernel can be written in the following form:

\[ T_{ij} = \sum_{m=1}^{12} (C_{m1} X_m n_1 + C_{m2} X_m n_2), \]  \hspace{1cm} (3.7)

where \( C_{mk} \) are combinations of the constants, and \( X_m \) is one of the terms \( (g_{ij}) \) defined in (A.4) in the Appendix. Thus we can write:

\[
\int_{\Delta L} T_{ij}(P, P_0) dL(P) = \sum_{\Delta m=1}^{12} (C_{m1} X_m n_1 + C_{m2} X_m n_2) dL(P)
= \sum_{m=1}^{12} (C_{m1} \int_{\Delta L} X_m n_1 dL(P) + C_{m2} \int_{\Delta L} X_m n_2 dL(P))
\]  \hspace{1cm} (3.8)

That is, the integration of \( T_{ij} \) is determined by the integrations of \( X_m n_1 \) and \( X_m n_2 \) on the element.
Apply quadratic line elements on the boundary and assume that the coordinates of the three
nodes on the element of integration are \((x_1, y_1), (x_2, y_2)\) (middle node) and \((x_3, y_3)\), and the
source point is \((x_0, y_0)\). Employing the quadratic shape functions:

\[
N_1(\xi) = (\xi - 1)\xi / 2, \quad N_2(\xi) = 1 - \xi^2, \quad N_3(\xi) = (\xi + 1)\xi / 2,
\]

with \(\xi\) being the natural coordinate, we have:

\[
r_1 = x - x_0 = \sum_{\alpha=1}^{3} N_\alpha x_\alpha - x_0 = \frac{x_1 + x_3 - 2x_2}{2} \xi^2 + \frac{x_3 - x_1}{2} \xi + x_2 - x_0,
\]

\[
r_2 = y - y_0 = \sum_{\alpha=1}^{3} N_\alpha y_\alpha - y_0 = \frac{y_1 + y_3 - 2y_2}{2} \xi^2 + \frac{y_3 - y_1}{2} \xi + y_2 - y_0. \tag{3.9}
\]

That is, \(r_1\) and \(r_2\) are quadratic functions of \(\xi\). On the element,

\[
n_1 = \frac{1}{J} \left[ (y_1 + y_3 - 2y_2)\xi + \frac{y_3 - y_1}{2} \right],
\]

\[
n_2 = -\frac{1}{J} \left[ (x_1 + x_3 - 2x_2)\xi + \frac{x_3 - x_1}{2} \right], \tag{3.10}
\]

\[dL = Jd\xi,
\]

where

\[
J = \sqrt{\left( \frac{dx}{d\xi} \right)^2 + \left( \frac{dy}{d\xi} \right)^2} = \sqrt{\left( x_1 + x_3 - 2x_2 \right)\xi^2 + \frac{x_3 - x_1}{2} \xi + \left( y_1 + y_3 - 2y_2 \right)\xi + \frac{y_3 - y_1}{2} \right]^2}
\]

is the Jacobian. Therefore, we obtain:

\[
\int_{\Delta t} X_m n_1 dL(P) = \frac{1}{\Delta t} X_m \left[ (y_1 + y_3 - 2y_2)\xi + \frac{y_3 - y_1}{2} \right] d\xi. \tag{3.11}
\]
Substituting the expressions (3.9) for $r_1$ and $r_2$ into $X_m$, which in turn is one of the terms in (A.4) in the Appendix, the above formula (3.11) can be shown to have the following final form:

$$
\int_0^1 \frac{a\xi^3 + b\xi^2 + c\xi + d}{e\xi^4 + f\xi^3 + g\xi^2 + h\xi + q} d\xi ,
$$

(3.12)

where constants $a$ to $q$ are combinations of the nodal coordinates and the material constants.

Note that the constants $e$, $f$, $g$, $h$ and $q$ in (3.12) cannot be zero at the same time unless $(x_1, y_1)$, $(x_2, y_2)$ (middle node) and $(x_3, y_3)$ are one point. The integration of $X_m n_2$ is handled in a similar way.

In general, an integral like (3.12) can be integrated analytically, given constants $a$ to $q$ (for example, using a symbolic manipulation software). Then integrals in (3.8) can be determined easily. In this way, we can convert the last integral in (3.5) containing the singular kernel to the summation of integrals containing polynomial fractions like (3.12), and these polynomial fraction integrals (3.12) do not depend on the integration path $\Delta L$. There is no difficulty at all in obtaining the exact value of such integrals, no matter how close the source point is to the element.

For the nearly weakly-singular integrals, that is, the one containing the $U_{ij}$ kernel in BIE (3.4) and the first integral on the right-hand side of (3.5), the nonlinear coordinate transformation developed in [18] is applied in the current study. It is found that this technique is equally effective and efficient in the case of the 2-D piezoelectric BIE.
3.4. Numerical Examples

To verify the procedures presented in the previous section for dealing with the nearly-
singular integrals in the piezoelectric BIE, a 2-D piezoelectric BEM program is developed and
two numerical test problems are studied in which the BEM solutions are obtained and
compared with the analytical ones when available.

3.4.1 Test problem 1: A thin piezoelectric film

First, a thin piezoelectric film under a uniform stretch \( \bar{u} \) in the \( x \) direction (Fig. 2) is
studied, for which analytical solutions can be found readily. We assume the dimension of the
film in \( z \) direction is very large so that it can be simplified as a plane strain problem. The
length \( L \) of the film is constant (= 1.0 m here), while the thickness \( h \) changes from \( L \) to \( 10^{-6} L \).
Note that the thickness \( h \) is changing from the macro- to micro-scales, relative to the length \( L \),
which may already be outside of the limits of the continuum mechanics assumptions for many
materials. However, it is of more interest here to verify the validity and effectiveness of
developed advanced BEM approach for such 2-D thin piezoelectric materials.

The boundary of the film is discretized with 10 quadratic boundary elements, with
four elements on each of the two horizontal (long) edges and one element on each of the two
vertical (short) edges. For the mechanical boundary conditions, displacement in \( y \) direction is
constrained along the edge \( y = 0 \); displacement in \( x \) direction is constrained along the edge
\( x = 0 \); and traction-free conditions are applied on other boundaries or directions (Fig. 2). For
the electric boundary conditions, the electric potential is zero along the top and bottom edges;
and the surface charge is zero on the two vertical edges. No body forces and changes are
applied to the piezoelectric body. The material used is PZT-4 and the material constants are:
as given in Ref. [27].

Figure 3.3 shows the results of stress $\sigma_x$ at the upper-right corner, for different thicknesses of the film, using the regular piezoelectric BEM without employing the techniques for dealing with the nearly-singular integrals. It is obvious that the BEM results deteriorate quickly as the thickness decreases (“m” in the plot represents the exponent in the thickness, e.g., “m = 1” means $h = 10^{-1}L$). Fig. 3.4 shows the same stress results by the advanced piezoelectric BEM using the techniques for dealing with the nearly-singular integrals, that is, the singularity subtraction and analytical evaluations, and the nonlinear coordinate transformation, as discussed in the previous section. We notice that even for the thickness-to-length ratio $h/L$ in the $10^{-6}$ range, the results are still very accurate and stable. The BEM results for the $x$-displacement of the point $(L/2, 0)$ (Fig. 3.5) are very accurate for this example, almost reproducing the exact values. The BEM results for the surface charge at the point $(L/2, 0)$ (Fig. 3.6) are also very close to the values from the analytical solution. All these BEM results demonstrate that the developed analytical methods for dealing with the nearly-singular integrals in the piezoelectric BEM procedure is very effective. It is also shown that the piezoelectric BIE does not degenerate when applied to thin shell-like piezoelectric structures as proved analytically in Ref. [43].

**3.4.2 Test problem 2: Piezoelectric coating on a rigid cylinder**

We next study the case of a rigid cylinder covered with a thin piezoelectric (PZT-4) coating (Fig. 3.7). The radius of the cylinder $a$ is fixed (= 1.0 m here), while the thickness of
the coating $h$ is reduced to test the developed piezoelectric BEM on this case with curved boundaries. For the mechanical boundary conditions, the coating is subjected to a uniform pressure load $p$ at the outer radius, and is fixed at the inner radius (interface with the cylinder). For the electric boundary conditions, zero surface charge is applied on both inner and outer boundaries. When the thickness $h$ decreases, the radial stress at the interface (e.g., point $A$, Fig. 3.7) should approach the value of the applied pressure, while the mechanical displacement at the interface should approach zero. These can be employed to verify the BEM solutions.

Two BEM meshes with very small numbers of elements are used for the coating, one with 8 quadratic elements (4 on each circular boundary) and another one with 16 quadratic elements (8 on each circle). Fig. 3.8 shows the results for the radial stress at point $A$ (interface) with the two boundary element discretizations, as the thickness of the PZT4 coating decreases from $h = 0.5a$, $10^{-1}a$, $10^{-2}a$, …, to $10^{-5}a$ (Again, “m” in the plot represents the exponent in the thickness). Both the BEM interface stress results approach the applied pressure as the thickness decreases, as expected. Fig. 3.9 gives the mechanical displacement (radial component) at point $B$, while Fig. 3.10 shows the electric displacement (tangential component) at point $B$. Both the mechanical and electric displacement components vanish with the decrease of the coating thickness, which is also expected.

This example demonstrates that the developed piezoelectric BEM for thin shell-like structures can handle models of curved boundary very efficiently and accurately as well. Stable and converged BEM results can be obtained with very few boundary elements, regardless how small the thickness of the piezoelectric coating is.
3.5. Conclusion

An advanced 2-D piezoelectric BEM has been developed for the analysis of thin shell-like piezoelectric structures or materials. A semi-analytical method has been devised for computing the nearly-singular integrals in the piezoelectric BIE for such applications.
Figure 3.1. A thin piezoelectric structure $V$ with boundary $S$.

Figure 3.2. A thin piezoelectric film.
Figure 3.3. Stress $\sigma_x$ at point $(L, h)$ with the regular piezoelectric BEM ($h/L = 10^{-m}$).

Figure 3.4. Stress $\sigma_x$ at point $(L, h)$ with the advanced piezoelectric BEM.
Figure 3.5. The $x$-component of displacement at point ($L/2$, 0).

Figure 3.6. The electric surface charge at point ($L/2$, 0).
Figure 3.7. A rigid cylinder with a thin piezoelectric coating under pressure $p$.

Figure 3.8. Magnitude of the stress $\sigma_x (\times p)$ at point $A$. 
Figure 3.9. Magnitude of the mechanical displacement $u_x (\times p)$ at point $B$.

Figure 3.10. Magnitude of the electric displacement $D_y (\times p)$ at point $B$. 
Chapter 4. Discussions

The numerical difficulty in the BEM modeling of thin structures is the nearly-singular integral problem. In this thesis, a method to deal with the nearly-singular integrals in the 2-D piezoelectric BIE (line integrals for 2-D problem) is developed, which converts these integrals into the summations of polynomial fraction integrals. A nonlinear coordinate transformation is also employed to increase the numerical accuracy. The method is found to be very accurate and efficient in dealing with nearly-singular integrals even if the ratio of the distance between the source point and the element to the element size is smaller than $10^{-6}$. For the test problems studied here, very promising results are obtained for thin structures with the aspect ratios in the micro-scale.

The key part of the developed method is the evaluation of (3.12), which is dealt with according to the roots of the denominator and then (3.12) is changed into two lower-rank polynomial fraction integrals and thus the analytical results are obtained. In fact, when the constant $a - f$ are all known in (3.12), it is very easy to obtain the results using a symbolic manipulation software such as Mathematica, that will be more reliable. How to use Mathematica in C++ code is another interesting topic for the nearly-singular integral computation in the BIE for piezoelectric thin structures.
Appendix A. The Fundamental Solutions

A.1 The U-Components in the 2-D Piezoelectric Fundamental Solution

We follow closely the results and notations in Refs. [25-27] for the piezoelectric fundamental solutions in the 2-D case. Introduce the following parameters (similar to $L_1$ to $L_6$ used in [25-27]):

\[
g_1 = \frac{\ln(p_0^2 r_1^2 + r_2^2)}{2},
\]

\[
g_2 = \frac{\ln[(p_1 r_1 + r_2)^2 + q_1^2 r_1^2]}{2},
\]

\[
g_3 = \frac{\ln[(p_1 r_1 - r_2)^2 + q_1^2 r_1^2]}{2},
\]

\[
g_4 = \arctan\left(\frac{r_2 + p_1 r_1}{q_1 r_1}\right),
\]

\[
g_5 = \arctan\left(\frac{r_2 - p_1 r_1}{q_1 r_1}\right),
\]

\[
g_6 = \arctan\left(\frac{r_2}{P_0 r_1}\right); \tag{A.1}
\]

and the parameters [25-27]:

\[
I_0 = R_0 g_1 + P_0 (g_2 + g_3) + Q_0 (g_4 - g_5),
\]

\[
I_1 = R_1 g_6 + P_1 (g_4 + g_5) + Q_1 (g_3 - g_2),
\]

\[
I_2 = R_2 g_1 + P_2 (g_2 + g_3) + Q_2 (g_4 - g_5), \tag{A.2}
\]
\[ I_3 = R_3 g_6 + P_3 (g_4 + g_5) + Q_3 (g_3 - g_2), \]

\[ I_4 = R_4 g_1 + P_4 (g_2 + g_3) + Q_4 (g_4 - g_5); \]

then the \(U\)-components in the 2-D piezoelectric fundamental solution can be expressed as:

\[ U_{11} = \frac{1}{2\pi} (\alpha_{11} I_4 + \beta_{11} I_2 + \gamma_{11} I_0), \]

\[ U_{21} = \frac{1}{2\pi} (\alpha_{12} I_3 + \beta_{12} I_1), \]

\[ U_{31} = -\frac{1}{2\pi} (\alpha_{13} I_3 + \beta_{13} I_1), \]

\[ U_{12} = U_{21}, \]

\[ U_{22} = \frac{1}{2\pi} (\alpha_{22} I_4 + \beta_{22} I_2 + \gamma_{22} I_0), \quad (A.3) \]

\[ U_{32} = -\frac{1}{2\pi} (\alpha_{23} I_4 + \beta_{23} I_2 + \gamma_{23} I_0), \]

\[ U_{13} = \frac{1}{2\pi} (\alpha_{13} I_3 + \beta_{13} I_1), \]

\[ U_{23} = \frac{1}{2\pi} (\alpha_{23} I_4 + \beta_{23} I_2 + \gamma_{23} I_0), \]

\[ U_{33} = -\frac{1}{2\pi} (\alpha_{33} I_4 + \beta_{33} I_2 + \gamma_{33} I_0), \]

where \(\alpha_{ij}, \beta_{ij}, \gamma_{ij}, R_i, P_i, Q_i, p_0, p_1, \) and \(q_1\) are parameters related to the material constants, as defined in Refs. [25-27]; and \(r_1\) and \(r_2\) are given in Eq. (17).
A.2 The T-Components in the 2-D Piezoelectric Fundamental Solution

The generalized traction kernels $T_{ij}$ are obtained by taking the derivatives of the generalized displacement kernel $U_{ij}$ and applying the relations as specified in Eqs. (3-8). First, introduce the following parameters (derivatives of $g_1, g_2, ..., g_6$ given in (A.1)):

$$g_{11} = \frac{p_0^2 r_i}{p_0^2 r_i^2 + r_2^2},$$

$$g_{12} = \frac{r_2}{p_0^2 r_i^2 + r_2^2},$$

$$g_{21} = \frac{(p_1^2 + q_1^2) r_i + p_1 r_2}{(p_1 r_i + r_2)^2 + q_1^2 r_i^2},$$

$$g_{22} = \frac{p_1 r_i + r_2}{(p_1 r_i + r_2)^2 + q_1^2 r_i^2},$$

$$g_{31} = \frac{(p_1^2 + q_1^2) r_i - p_1 r_2}{(p_1 r_i - r_2)^2 + q_1^2 r_i^2},$$

$$g_{32} = \frac{r_2 - p_1 r_i}{(p_1 r_i - r_2)^2 + q_1^2 r_i^2},$$

$$g_{41} = \frac{-q_1 r_2}{(p_1 r_i + r_2)^2 + q_1^2 r_i^2},$$

$$g_{42} = \frac{q_1 r_i}{(p_1 r_i + r_2)^2 + q_1^2 r_i^2},$$

$$g_{51} = \frac{-q_1 r_2}{(p_1 r_i - r_2)^2 + q_1^2 r_i^2},$$

$$g_{52} = \frac{q_1 r_i}{(p_1 r_i - r_2)^2 + q_1^2 r_i^2},$$

$$g_{61} = \frac{-q_1 r_2}{(p_1 r_i - r_2)^2 + q_1^2 r_i^2},$$

$$g_{62} = \frac{q_1 r_i}{(p_1 r_i - r_2)^2 + q_1^2 r_i^2}.$$
\[ g_{61} = \frac{-p_0 r_2}{p_0 r_1^2 + r_2^2}, \]
\[ g_{62} = \frac{p_0 r_1}{p_0 r_1^2 + r_2^2}, \]

and

\[ I_{01} = R_0 g_{11} + P_0 (g_{21} + g_{31}) + Q_0 (g_{41} + g_{51}), \]
\[ I_{11} = R_1 g_{61} + P_1 (g_{41} + g_{51}) + Q_1 (g_{31} + g_{21}), \]
\[ I_{21} = R_2 g_{11} + P_2 (g_{21} + g_{31}) + Q_2 (g_{41} + g_{51}), \]
\[ I_{31} = R_3 g_{61} + P_3 (g_{41} + g_{51}) + Q_3 (g_{31} - g_{21}), \]
\[ I_{41} = R_4 g_{11} + P_4 (g_{21} + g_{31}) + Q_4 (g_{41} - g_{51}), \]
\[ I_{02} = R_0 g_{12} + P_0 (g_{22} + g_{32}) + Q_0 (g_{42} - g_{52}), \]
\[ I_{12} = R_1 g_{62} + P_1 (g_{42} + g_{52}) + Q_1 (g_{32} - g_{22}), \]
\[ I_{22} = R_2 g_{12} + P_2 (g_{22} + g_{32}) + Q_2 (g_{42} - g_{52}), \]
\[ I_{32} = R_3 g_{62} + P_3 (g_{42} + g_{52}) + Q_3 (g_{32} - g_{22}), \]
\[ I_{42} = R_4 g_{12} + P_4 (g_{22} + g_{32}) + Q_4 (g_{42} - g_{52}); \]

and finally

\[ U_{111} = \frac{1}{2\pi} (\alpha_{11} I_{41} + \beta_{11} I_{21} + \gamma_{11} I_{01}), \]
\[ U_{211} = \frac{1}{2\pi} (\alpha_{12} I_{31} + \beta_{12} I_{11}), \]
\[ U_{311} = -\frac{1}{2\pi} (\alpha_{13} I_{31} + \beta_{13} I_{11}) , \]

\[ U_{121} = U_{211} , \]

\[ U_{221} = \frac{1}{2\pi} (\alpha_{22} I_{41} + \beta_{22} I_{21} + \gamma_{22} I_{01}) , \]

\[ U_{321} = -\frac{1}{2\pi} (\alpha_{23} I_{41} + \beta_{23} I_{21} + \gamma_{23} I_{01}) , \]

\[ U_{131} = \frac{1}{2\pi} (\alpha_{13} I_{31} + \beta_{13} I_{11}) , \]

\[ U_{231} = \frac{1}{2\pi} (\alpha_{23} I_{41} + \beta_{23} I_{21} + \gamma_{23} I_{01}) , \]

\[ U_{331} = -\frac{1}{2\pi} (\alpha_{33} I_{41} + \beta_{33} I_{21} + \gamma_{33} I_{01}) , \]

(A.6)

\[ U_{112} = \frac{1}{2\pi} (\alpha_{11} I_{42} + \beta_{11} I_{22} + \gamma_{11} I_{02}) , \]

\[ U_{212} = \frac{1}{2\pi} (\alpha_{12} I_{32} + \beta_{12} I_{12}) , \]

\[ U_{312} = -\frac{1}{2\pi} (\alpha_{13} I_{32} + \beta_{13} I_{12}) , \]

\[ U_{122} = U_{212} , \]

\[ U_{222} = \frac{1}{2\pi} (\alpha_{22} I_{42} + \beta_{22} I_{22} + \gamma_{22} I_{02}) , \]

\[ U_{322} = -\frac{1}{2\pi} (\alpha_{23} I_{42} + \beta_{23} I_{22} + \gamma_{23} I_{02}) , \]
\[ U_{132} = \frac{1}{2\pi}(\alpha_{13}I_{32} + \beta_{13}I_{12}), \]

\[ U_{232} = \frac{1}{2\pi}(\alpha_{23}I_{42} + \beta_{23}I_{22} + \gamma_{23}I_{10}), \]

\[ U_{332} = -\frac{1}{2\pi}(\alpha_{33}I_{42} + \beta_{33}I_{22} + \gamma_{33}I_{10}); \]

then the \( T \)-components in the 2-D piezoelectric fundamental solution can be written as:

\[ T_{11} = \left(c_{11}U_{111} + c_{12}U_{122} + e_{21}U_{132}\right)n_1 + \left(c_{33}(U_{112} + U_{121}) + e_{13}U_{131}\right)n_2, \]

\[ T_{21} = \left(c_{11}U_{211} + c_{12}U_{222} + e_{21}U_{232}\right)n_1 + \left(c_{33}(U_{212} + U_{221}) + e_{13}U_{231}\right)n_2, \]

\[ T_{31} = \left(c_{11}U_{311} + c_{12}U_{322} + e_{21}U_{332}\right)n_1 + \left(c_{33}(U_{312} + U_{321}) + e_{13}U_{331}\right)n_2, \]

\[ T_{12} = \left(c_{12}U_{111} + c_{22}U_{122} + e_{22}U_{132}\right)n_2 + \left(c_{33}(U_{112} + U_{121}) + e_{13}U_{131}\right)n_1, \]

\[ T_{22} = \left(c_{12}U_{211} + c_{22}U_{222} + e_{22}U_{232}\right)n_2 + \left(c_{33}(U_{212} + U_{221}) + e_{13}U_{231}\right)n_1, \]

\[ T_{32} = \left(c_{12}U_{311} + c_{22}U_{322} + e_{22}U_{332}\right)n_2 + \left(c_{33}(U_{312} + U_{321}) + e_{13}U_{331}\right)n_1, \]

\[ T_{13} = -(e_{13}(U_{112} + U_{121}) - e_{11}U_{131})n_1 - (e_{21}U_{111} + e_{22}U_{122} - e_{22}U_{132})n_2, \]

\[ T_{23} = -(e_{13}(U_{212} + U_{221}) - e_{11}U_{231})n_1 - (e_{21}U_{211} + e_{22}U_{222} - e_{22}U_{232})n_2, \]

\[ T_{33} = -(e_{13}(U_{312} + U_{321}) - e_{11}U_{331})n_1 - (e_{21}U_{311} + e_{22}U_{322} - e_{22}U_{332})n_2, \]

where the material constants \( c_{11} \) to \( e_{22} \) are as defined in Refs. [25-27], and \( n_1 \) and \( n_2 \) are the directional cosines of the normal \( n \).
class TdPiezoBem: public TdBem
{
    
    public:

    //Constructions and the Destruction.
    TdPiezoBem();

    TdPiezoBem(double cc11, double cc12, double cc22, double cc33, double ee21, double ee22, double ee13, double ss11, double s22,
               Array<BoundaryNode*>* pNodArray, Array<IsoPameElement*>* pElementArray,
               Array<BoundaryCondition>* BouConArray, BOOL IsInfiniteDomain);

    ~TdPiezoBem();

    //Here are some functions about the basic Parameters.
    void SetMaterialParameter (double c11, double c12, double c22, double c33, double e21, double e22, double e13, double s11, double s22);

    double GetParameterC11() const { return c11; };

    double GetParameterC12() const { return c12; };

    double GetParameterC22() const { return c22; };

    double GetParameterC33() const { return c33; };

    double GetParameterE21() const { return e21; };

    double GetParameterE22() const { return e22; };

    double GetParameterE13() const { return e13; };

}
double GetParameterS11() const {return s11;};

double GetParameterS22() const {return s22;};

    //Here are functions for the Node, the Element and Condition.
void SetNodePointerArray(Array<BoundaryNode*>* pBNodeArray);
void SetElementPointerArray(Array<IsoPameElement*>* pBElementPointerArray);
void SetBoundaryConditionArray(Array<BoundaryCondition>* pBConditionArray);

    // For the input and output files.
virtual BOOL ReadInputFile(char* filename);
virtual BOOL WriteOutputFile(char* filename);

    // The derived Parameters.
void Setqpp();

    // Computing part.
virtual void ComputeKernelwithSingularity(const BoundaryNode& Node,
    IsoPameElement*& pElement,
    Array<Matrix<double>>& Ker_G, Array<Matrix<double>>& Ker_H);  
virtual void ComputeKernelwithoutSingularity(const BoundaryNode& Node,
    IsoPameElement*& pElement,
    Array<Matrix<double>>& Ker_G, Array<Matrix<double>>& Ker_H);  

virtual Matrix<double> FirstFundSolution(const Vector& SourceVector, const Vector& EndVector);
Matrix<double> CoefMatrix();
virtual Matrix<double> SecondFundSolution(const Vector& SourceVector, const Vector& EndVector, const Vector& Normal);

Matrix<double> CoefMatrixforT(double x0, double x1, double x2, double x3, double y0, double y1, double y2, double y3, IsoPameElement*& pElement);

double Integration0(double a, double b, double c, double d, double e, double f, double g, double md, double me, IsoPameElement*& pElement);

double Integration(double a, double b, double c, double d, double e, double f, double g, double md, double me);

double Integration3(double a, double b, double c, double d, double e, double f, double g, double me);

double Integration2(double b, double c, double d, double e, double f, double g, double me);

int equal(double t1, double t2, double t3, double t4);

void OpenNearSingularitySwitch();

Vector NormalStress(int i, Array<double>& Displacement, Array<double>& Traction, Matrix<double>& NormalStress);

private:

    static double PI;

    static const double RatioforNearSingularity;

    static const int NonLinearTransformOrder;

    static Array<double> NonLinearValue;

    //c --- Elasticity moduli measured at constant or zero electric field.

    //e --- Piezoelectric tensor.
//s --- Dielectric tensor measured at constant or zero strain.

double c11,c12,c22,c33;
double e21,e22,e13;
double s11,s22;

//Derived Parameters.

double p0,q1,p1;

Array<BoundaryNode*>* pBNodePointerArray;
Array<IsoPameElement*>* pBElementPointerArray;
Array<BoundaryCondition>* pBConditionArray;

ifstream InputFile;
ofstream OutputFile;

BOOL NearSingularConsidered;

void Free();

};
References


