An Introduction to the Boundary Element Method (BEM) and Its Applications in Engineering

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(Updated: November 8, 2013)
Boundary Element Method (BEM)

- **Boundary element method** applies surface elements on a 3-D domain and line elements on a 2-D domain. Number of elements is $O(n^2)$ as compared to $O(n^3)$ for other domain based methods ($n = \text{number of elements/dimension}$).

- BEM is good for problems with complicated geometries, stress concentration problems, infinite domain problems, wave propagation problems, and many others.

- **Finite element method** can now solve a model with 1 million DOFs on a PC with 1GB RAM.

- **Fast multipole BEM** can also solve a model with 1 million DOFs on a PC with 1 GB RAM. However, these DOFs are on the boundary of the model only, which would require 1 billion DOFs for a corresponding domain model.
Two Different Approaches in Computational Mechanics

Engineering Problems

Mathematical Models

Differential Equation (ODE/PDE) Formulations

Boundary Integral Equation (BIE) Formulations

Analytical Solutions

Numerical Solutions

Analytical Solutions

Numerical Solutions

FDM

FEM

EFM

Others

BEM

BNM

Others
A Brief History of the BEM

Integral equations (Fredholm, 1903)
- 2D Potential Problems

Jaswon and Symm (1963)
- 2D Potential Problems

T. A. Cruse and F. J. Rizzo (1968)
- 2D Elastodynamics

BEM emerged in 1980’s …

Modern numerical solutions of BIEs (in early 1960’s)

P. K. Banerjee (1975)
- Coined the name “boundary element method”

F. J. Rizzo (Dissertation in 1964 at TAM UIUC, paper in 1967)
- 2D Elasticity Problems
A Comparison of the FEM and BEM - An Engine Block Model

- Heat conduction of a V6 engine model is studied.
- ANSYS is used in the FEM study.
- Fast multipole BEM is used in the BEM study.
- A linear temperature distribution is applied on the six cylindrical surfaces

FEM (363,180 volume elements)
BEM (42,169 surface elements)
A Comparison of the FEM and BEM with An Engine Block Model (Cont.)

FEM Results (50 min.)

BEM Results (16 min.)
Formulation: The Potential Problem

- Governing Equation
  \[ \nabla^2 u(x) = 0, \quad \forall x \in V; \]
  with given boundary conditions on \( S \)
- The Green’s function for potential problem
  \[ G(x, y) = \frac{1}{2\pi} \ln \left( \frac{1}{r} \right), \quad \text{in 2D}; \]
  \[ G(x, y) = \frac{1}{4\pi r}, \quad \text{in 3D}. \]
- Boundary integral equation formulation
  \[ C(x)u(x) = \int_S \left[ G(x, y)q(y) - F(x, y)u(y) \right] ds(y), \quad \forall x \in V \text{ or } S, \]
  where \( q = \partial u / \partial n, \quad F = \partial G / \partial n. \)
- Comments: The BIE is exact due to the use of the Green’s function;
  Note the singularity of the Green’s function \( G(x, y). \)
Formulation: The Potential Problem (Cont.)

• Discretize boundary $S$ using $N$ boundary elements:
  - line elements for 2D problems;
  - surface elements for 3D problems.

• The BIE yields the following BEM equation

$$
\begin{bmatrix}
  f_{11} & f_{12} & \cdots & f_{1N} \\
  f_{21} & f_{22} & \cdots & f_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{N1} & f_{N2} & \cdots & f_{NN}
\end{bmatrix}
\begin{bmatrix}
  q_1 \\
  q_2 \\
  \vdots \\
  q_N
\end{bmatrix}
=
\begin{bmatrix}
  g_{11} & g_{12} & \cdots & g_{1N} \\
  g_{21} & g_{22} & \cdots & g_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{N1} & g_{N2} & \cdots & g_{NN}
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_N
\end{bmatrix}
$$

• Apply the boundary conditions to obtain

$$
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1N} \\
  a_{21} & a_{22} & \cdots & a_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{N1} & a_{N2} & \cdots & a_{NN}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N
\end{bmatrix}
=
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_N
\end{bmatrix}
$$

or

$$\mathbf{Ax} = \mathbf{b}$$

Each node/element interacts with all other node/element directly.

The number of operations is of order $O(N^2)$.

Storage is also of order $O(N^2)$. 
Advantages and Disadvantages of the BEM

Advantages:
  • Accuracy – due to the semi-analytical nature and use of integrals.
  • More efficient in modeling due to the reduction of dimensions.
  • Good for stress concentration and infinite domain problems.
  • Good for modeling thin shell-like structures/materials.
  • Neat … (integration, superposition, boundary solutions for BVPs).

Disadvantages:
  • Conventional BEM matrices are dense and nonsymmetrical.
  • Solution time is long and memory size is large (Both are $O(N^2)$).
  • Limited to solving small-scale models (Not any more with new fast solution methods).
Fast Multipole Method (FMM)

- FMM can reduce the cost (CPU time & storage) for BEM to $O(N)$
- Pioneered by Rokhlin and Greengard (mid of 1980’s)
- Ranked among the top ten algorithms of the 20th century (with FFT, QR, …) in computing
- An earlier review by Nishimura: *Applied Mechanics Review*, July 2002
The Simple Idea

Apply iterative solver (GMRES) and accelerate matrix-vector multiplications by replacing element-element interactions with cell-cell interactions.

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1N} \\
  a_{21} & a_{22} & \cdots & a_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{N1} & a_{N2} & \cdots & a_{NN}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N
\end{bmatrix} =
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_N
\end{bmatrix}, \quad \text{or} \quad Ax = b
\]

Conventional BEM approach \((O(N^2))\)  \hspace{1cm} FMM BEM approach \((O(N) \text{ for large } N)\)
Adaptive Cross Approximation (ACA)

- Hierarchical decomposition of a BEM matrix:

\[ \begin{array}{c|cc}
73 & 38 & 9 \\
38 & 9 & 38 \\
\end{array} \]

\[ \begin{array}{c|cc}
20 & 8 & 9 \\
8 & 20 & 9 \\
\end{array} \]

\[ \begin{array}{c|cc}
12 & 7 & 8 \\
7 & 12 & 8 \\
\end{array} \]

(from Rjasanow and Steinbach, 2007)

- A lower-rank submatrix $A$ away from the main diagonal can be represented by a few selected columns ($u$) and rows ($v^T$) (crosses) based on error estimates:

\[ A_k \approx \sum_{\alpha=1}^{k} \frac{1}{\gamma_{\alpha}} u_{\alpha} v_{\alpha}^T, \text{ with } \gamma = A(i,j), \ u = A(:,j), \ v = A(i,:). \]

- The process is independent of the kernels (or 2-D/3-D)
- Can be integrated with iterative solvers (GMRES)
Some Applications of the Fast Multipole Boundary Element Method

- 2-D/3-D potential problems.
- 2-D/3-D elasticity problems.
- 2-D/3-D Stokes flow problems.
- 2-D/3-D acoustics problems.
- Applications in modeling porous materials, fiber-reinforced composites and microelectromechanical systems (MEMS).
- All software packages used here can be downloaded from www.yijunliu.com.
2-D Potential: Accuracy and Efficiency of the Fast Multipole BEM

Results for a simple potential problem in an annular region $V$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$q_a$ ($FMM BEM$)</th>
<th>$q_a$ ($Conventional BEM$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>-401.771619</td>
<td>-401.771546</td>
</tr>
<tr>
<td>72</td>
<td>-400.400634</td>
<td>-400.400662</td>
</tr>
<tr>
<td>360</td>
<td>-400.014881</td>
<td>-400.014803</td>
</tr>
<tr>
<td>720</td>
<td>-400.003468</td>
<td>-400.003629</td>
</tr>
<tr>
<td>1440</td>
<td>-400.000695</td>
<td>-400.000533</td>
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<tr>
<td>2400</td>
<td>-400.001929</td>
<td>-400.000612</td>
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<tr>
<td>4800</td>
<td>-400.001557</td>
<td>-400.000561</td>
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<tr>
<td>7200</td>
<td>-399.997329</td>
<td>-399.998183</td>
</tr>
<tr>
<td>9600</td>
<td>-399.997657</td>
<td>-399.996874</td>
</tr>
</tbody>
</table>

Analytical Solution: -400.0
3-D Potential: Modeling of Fuel Cells

Thermal Analysis of Fuel Cell (SOFC) Stacks

There are 9,000 small side holes in this model

Total DOFs = 530,230, solved on a desktop PC with 1 GB RAM)

ANSYS can only model one cell on the same PC
3-D Electrostatic Analysis

- 11 conducting spheres.
- Forces can be found with the charge density.
- Largest model has 118,800 DOFs.

One BEM mesh

Computed charge density
3-D Electrostatic Analysis (Cont.)

Applications in MEMS

- Beams are applied with +/- voltages.
- Forces can be found with the charge density.
- Model shown has 55 beams (179,300 DOFs).

A comb drive
Mechanical Engineering
2-D Elasticity: Modeling of Perforated Plates

A BEM model of a perforated plate (with 1,600 holes)

Computed effective Young’s modulus for the perforated plate (\(x\ E\))

<table>
<thead>
<tr>
<th>No. Holes</th>
<th>DOFs</th>
<th>Uniformly Distributed Holes</th>
<th>Randomly Distributed Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x2</td>
<td>3,680</td>
<td>0.697892</td>
<td>0.678698</td>
</tr>
<tr>
<td>4x4</td>
<td>13,120</td>
<td>0.711998</td>
<td>0.682582</td>
</tr>
<tr>
<td>6x6</td>
<td>28,320</td>
<td>0.715846</td>
<td>0.659881</td>
</tr>
<tr>
<td>8x8</td>
<td>49,280</td>
<td>0.717643</td>
<td>0.651026</td>
</tr>
<tr>
<td>12x12</td>
<td>108,480</td>
<td>0.719345</td>
<td>0.672084</td>
</tr>
<tr>
<td>20x20</td>
<td>296,000</td>
<td>0.720634</td>
<td>0.676350</td>
</tr>
<tr>
<td>30x30</td>
<td>660,000</td>
<td>0.721255</td>
<td>0.676757</td>
</tr>
<tr>
<td>40x40</td>
<td>1,168,000</td>
<td>0.721558</td>
<td>0.675261</td>
</tr>
</tbody>
</table>
3-D Elasticity: Modeling of Scaffold Materials

(Hollister, et al, 2002)
2-D Stokes Flow: Multiple Cylinders

(c) A larger model with $13 \times 13$ elliptic cylinders and $a = 0.02h$, $b = 0.5a$, DOFs = 103,000.
3-D Stokes Flow: Modeling of RBCs

An exterior Stokes flow problem.
Total DOFs = 900 K; Solved on a laptop PC.

Drag force in the flow direction.
3-D Stokes Flow: MEMS Analysis

• BEM model with 362,662 elements (1,087,986 total DOFs)
• An angular velocity is applied
• Drag forces are computed
• Solved on a desktop PC
Modeling CNT Composites

(a) An RVE with many CNT fibers (to be solved by the fast multipole BEM)

(b) Models for the CNTs and interfaces (to be extracted from MD simulations)
A Multiscale Model for CNT Composites

• A rigid-inclusion model is applied to represent the CNT fibers in polymer matrix.
• The cohesive model from MD study is applied for the CNT/polymer interfaces.

A cohesive interface model:

\[ u^{(CNT)} - u = C \mathbf{t}, \quad \forall y \in S_\alpha, \]

with \( C \) being the compliance matrix (determined by MD).

• The fast multipole BEM is applied to solve the large BEM systems.
• This approach is a first step toward the more general multiscale model with continuum BEM for matrix, and nanoscale MD for CNTs and interfaces.
A Typical RVE Using the BEM

A model containing 2,197 short CNT fibers with the total DOF = 3,018,678
An RVE containing 2,000 CNT fibers with the total DOF = 3,612,000 (CNT length = 50 nm, volume fraction = 10.48%). A larger model with 16,000 CNT fibers (8 times of what is shown above) and 28.9M DOFs was solved successfully on a FUJITSU HPC2500 supercomputer at Kyoto University.
Modeling of CNT Composites (Cont.)
Effects of the Cohesive Interface

Computed effective moduli of CNT/polymer composites
(same CNT and RVE dimensions as used in the previous perfect bonding case)

Case 1:
C11=C22=C33=0
(perfect bonding)

Case 2:
C11=C22=C33=Cr = 0.02157 (large stiffness)

Case 3:
C11=C22=C33=Cz = 3.506 (small stiffness)

Cr, Cz are interface compliance ratios in the radial and longitudinal direction of the fiber, respectively, and are determined from the MD simulations.

Closer to experimental data
Acoustic Wave Problems

- Helmholtz equation:
  \[ \nabla^2 \phi + k^2 \phi + Q\delta(x, x_0) = 0, \quad \forall x \in E \]
- \( \phi \) - acoustic pressure, \( k = \omega / c \) - wavenumber
- BEM for solving 3-D full-/half-space, interior/exterior, radiation/scattering problems

Infinite half-space/symmetry plane (no elements are needed)
Examples: A Radiating Sphere
$O(N)$ Computing Efficiencies

![Graph showing CPU time vs. DOFs for different BEM methods]

- **Fast multipole BEM (ka = 2)**
- **ACA BEM (ka = 2)**
- **Conventional BEM (ka = 2)**
- **Fast multipole BEM (ka = 20)**
- **ACA BEM (ka = 20)**
- **Conventional BEM (ka = 20)**

**CPU Time (sec.)** vs. **DOFs**
Windmill Turbine Analysis

Plot of the SPL on the field due to 5 windmills (with 557,470 DOFs)
FEM/BEM Coupled Analysis (Freq. Response)
Noise Prediction in Airplane Landing/Taking Off

Noise propagation on the ground during the landing of an airplane, BEM model with 539,722 elements and solved with the FMM BEM in 8940 sec on a PC ($ka = 61.5$ or $f = 90$ Hz).
Acoustic Noise During Launch of A Space Vehicle

- Jet flow was modeled using CFD by NASA
- Acoustic field was modeled using our acoustic fast BEM code
- FFT used to compute the time domain solutions
- The BEM model with 300K elements was solved on a laptop PC.
Bio-Medical Applications

A human head model with 90,000 elements

Pressure plots at 11 kHz with a plane wave in \( -x \) direction
Bio-Medical Applications (Cont.)
Applications in Computer Animation

Work done by the Group of Professor Doug James at Cornell University, Using the *FastBEM Acoustics* code

(Click on the images to play the YouTube video and *hear* the *computed* sound)
Fast Multipole Boundary Element Method (*FastBEM*) Software for Education, Research and Further Development

([http://urbana.mie.uc.edu/yliu/Software](http://urbana.mie.uc.edu/yliu/Software))
Summary

- BEM is very efficient for solving large-scale problems with complicated geometries or in infinite domains.
- Fast multipole method has re-energized the BEM research and dramatically expanded its range of applications.
- More large-scale, realistic engineering problems can be, and should be, solved by the fast multipole BEM.
- Other developments in fast multipole BEM: fracture mechanics, elastodynamic and electromagnetic wave propagation problems, time-domain problems, black-box fast multipole method (bbFMM), coupled field and nonlinear problems.
- Other fast solution methods for solving BIE/BEM equations include: adaptive cross approximation (ACA) method, precorrected FFT method, wavelet method, and others.
A Bigger Picture of the CM
– A Numerical Toolbox

**FEM**: Large-scale structural, nonlinear, and transient problems

**BEM**: Large-scale continuum, linear, and steady state (wave) problems

**Meshfree**: Large deformation, fracture and moving boundary problems

“If the only tool you have is a hammer, then every problem you can solve looks like a nail!”


Acknowledgments

• The US National Science Foundation
• NASA
• Prof. Subrata Mukherjee at Cornell University
• Prof. Naoshi Nishimura at Kyoto University (Japan)
• Prof. Dong Qian at the University of Cincinnati
• Students at the University of Cincinnati and Kyoto University
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